

# Simple Equilibria in General Contests\*

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## Abstract

We show how symmetric equilibria emerge in general two-player contests in which skill and effort are combined to produce output according to a general production technology and players have skills drawn from different distributions. The model includes the Tullock (1980) and Lazear and Rosen (1981) models as special cases. Our paper provides intuition regarding how the contest components interact to determine the incentive to exert effort and sheds new light on classic comparative statics results. In particular, we show that more heterogeneity can increase equilibrium effort.

**Keywords:** contest theory, symmetric equilibrium, heterogeneity, risk, stochastic dominance

**JEL classification:** C72, D74, D81, J23, M51

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# 1 Introduction

In a contest, players invest effort or other costly resources to win a prize. Many economic interactions can be modeled as a contest. Promotions, for example, represent an important incentive in many firms and organizations. Employees exert effort to perform better than their colleagues and, thus, to be considered for promotion to a more highly paid position. Litigation can also be understood as a contest, in which the different parties spend time and resources to prevail in court. Procurement is a third example, where different firms invest resources into developing a proposal or lobbying politicians, thereby increasing the odds of being selected, receiving some rent in return.

Players participating in contests are typically heterogeneous in some respect. For instance, employees differ with respect to their skills, the litigant parties differ with respect to the quality of the available evidence, and firms differ with respect to their capabilities of designing a proposal. When accounting for such heterogeneity in contest models, equilibria are often asymmetric, meaning that players choose different levels of effort. Due to this asymmetry, to keep the analyses tractable, researchers have often imposed rather strict assumptions regarding the production technology and the distributions of stochastic components of the contest.

In this paper, we consider a general contest model that allows players to be heterogeneous in terms of their skill distributions. We show that, nevertheless, there is generally a symmetric equilibrium in which players choose the same effort. In our contest model, the output of each player, and thereby the player's *production* or *contribution* to the contest, is determined by a general function of individual effort and the realization of a random variable. We refer to the random variable as the skill of the player (typically, and equivalently, referred to as noise in the contest theory literature) and the statistical distributions of possible skill realizations are allowed to be different for the competing players. The player with the highest output wins the contest. The model is general in terms of production functions and skill distributions and includes the well-known models by Tullock (1980) and Lazear and Rosen (1981) as special cases. The skill distributions of the competing players (including the expected values) are assumed to be common knowledge, whereas the exact skill realizations are generally (symmetrically) unknown (as, e.g., in Holmström 1982). These assumptions realistically reflect that in a promotion contest, for example, the expected skill of a player may be commonly known (e.g., the education, prior work experience, or CV of a player), whereas the exact skill level for the particular job is unknown (e.g., there might be uncertainty regarding how education

translates into workplace performance and job match).

We make two primary contributions. The first contribution is to show the existence of a symmetric (pure-strategy equal-effort) equilibrium in a general two-player contest setting where players have heterogeneous skill distributions. Moreover, we provide intuition regarding how the different components of the contest interact to determine the incentive to exert effort. More specifically, we highlight the interaction between three factors. The first factor relates to the production technology and is the ratio of the marginal product of effort and the marginal product of skill. The intuition behind this factor is that the purpose of a marginal effort increase for an individual player is to beat marginally more able rivals. The ratio describes how effective a marginal effort increase is to overcome the output advantage of marginally more skilled players. The second and third factors are represented by the product of the densities of the skill distributions of the two competing players, evaluated at the same point. The reason for the presence of this product is that a marginal effort increase is pivotal to winning the contest only in cases where the skill realizations of the two players are exactly the same, and the product describes the “likelihood” of this event to happen.

The second contribution is that we use the simple structure of the equilibrium to construct a link between our contest model and standard models of decision-making under risk (expected utility theory), allowing us to revisit important comparative statics results of contest theory. In particular, we analyze how equilibrium effort is affected by making the skill distributions of the competing players more heterogeneous, investigating both the role of differences in expected skill (conceptualized by first-order stochastic dominance) and the role of differences in the uncertainty of the skill distributions (conceptualized by mean-preserving spreads), and how these relationships are affected by the production technology. The general message is that making contest participants more heterogeneous can increase equilibrium effort. These findings contradict certain “standard” results known from the Tullock contest and the Lazear-Rosen tournament. Thus, the comparative statics results derived from those standard models are not representative of the conclusions derived in the more general model.

The paper is organized as follows. In Section 2 below, we discuss related literature. Section 3 introduces the contest model. Section 4 provides the equilibrium characterization. In Section 5, we present comparative statics results. Finally, Section 6 concludes.

## 2 Related literature

There are three main approaches to the study of contests, the Tullock (or ratio-form) contest, the Lazear-Rosen tournament, and the complete-information all-pay auction.<sup>1</sup> In the Tullock contest, introduced by Tullock (1980), a player's winning probability is given by his or her contribution to the contest divided by the sum of the contributions of the competing players, and the contribution of each player is typically defined as a function of effort and sometimes also of skill.<sup>2</sup> The Lazear-Rosen tournament assumes that the player with the highest contribution wins with certainty, and contributions depend on effort, some random factors (e.g., luck), and possibly on skills. The seminal paper is by Lazear and Rosen (1981) who apply the model in a labor-market context.<sup>3</sup> The all-pay auction, finally, makes the same assumption as the Lazear-Rosen tournament, except that contest contributions are deterministic and do not depend on random factors.<sup>4</sup>

Most studies analyzing the Tullock contest and the Lazear-Rosen tournament impose assumptions that ensure that equilibria in pure strategies exist. In contrast, only mixed-strategy equilibria exist in the all-pay auction (when players are symmetrically informed about the decision situation). The Tullock contest and the Lazear-Rosen tournament are special cases of our model, while the all-pay auction is not. Our main contribution is to show that the simple structure of equilibria in the Tullock contest and the Lazear-Rosen tournament extends to more general production functions and skill distributions, even if players are heterogeneous in terms of their skill distributions.<sup>5</sup> To our knowledge, our paper is the first in the contest literature to consider general production functions; the

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<sup>1</sup>The theoretical contest literature has been surveyed in a number of books and papers. See, e.g., Konrad (2009) and Vojnović (2016) for recent textbooks and Chowdhury and Gürtler (2015), Chowdhury, Esteve-Gonzalez, and Mukherjee (2019), and Fu and Wu (2019) for recent surveys.

<sup>2</sup>The Tullock contest has been analyzed by, e.g., Hillman and Riley (1989), Cornes and Hartley (2005), Fu and Lu (2009a,b), Corchón and Dahm (2010), Schweinzer and Segev (2012), and Chowdhury and Kim (2017). It has been axiomatized in various settings by Skaperdas (1996), Clark and Riis (1998b), and Münster (2009).

<sup>3</sup>It has been further analyzed by, for example, Green and Stokey (1983), Malcomson (1984, 1986), O'Keefe, Viscusi, and Zeckhauser (1984), Lazear (1989), Schotter and Weigelt (1992), Zájbojník and Bernhardt (2001), Hvide (2002), Grund and Sliwka (2005), Schöttner and Thiele (2010), Aoyagi (2010), Gürtler and Gürtler (2015), and Imhof and Kräkel (2016).

<sup>4</sup>A detailed equilibrium characterization of the all-pay auction was developed by Baye, Kovenock, and de Vries (1996). The complete-information all-pay auction (with mixed-strategy equilibria) is the most commonly used in contest theory, but a private-values version can be found as well. The all-pay auction has been further studied by, e.g., Clark and Riis (1998a), Barut and Kovenock (1998), Moldovanu and Sela (2001, 2006), Moldovanu, Sela, and Shi (2007), Cohen, Kaplan, and Sela (2008), Siegel (2009, 2010), Sela (2012), Morath and Münster (2013), Barbieri, Malueg, and Topolyan (2014), Pérez-Castrillo and Wettstein (2016), Olszewski and Siegel (2016), and Fang, Noe, and Strack (2020).

<sup>5</sup>Kirkegaard (2022) has recently proposed a contest model similar to ours. However, as his focus is on optimal contest design, we view his work as complementary to ours.

previous literature has focused on the case of linear or multiplicative technologies.

An additional contribution of our paper is to show that canonical results regarding how player heterogeneity affects equilibrium effort in the Tullock contest and the Lazear-Rosen tournament do not always extend to more general production functions and skill distributions.<sup>6</sup> For example, Schotter and Weigelt (1992) have shown that effort is higher when players have homogeneous skills relative to when they are heterogeneous, since disadvantaged players tend to give up and reduce their effort, whereas advantaged players can afford to reduce their effort. Moreover, several studies (e.g., Hvide 2002) have shown that greater uncertainty regarding the contest outcome tends to reduce effort as, intuitively, effort has a lower impact on who becomes the winner in a contest where the outcome is heavily influenced by random factors. In our setting, the above results can be overturned, as we find that greater heterogeneity in terms of the skill distributions of the competing players and more uncertainty regarding the contest outcome can in many cases result in higher equilibrium effort. Drugov and Ryvkin (2021) complement these findings as they also demonstrate that the “encouragement effect” can be a general result, contrary to long-held views in the contest theory literature. In contrast to the micro-foundation approach in the present paper, they employ a reduced-form analysis working with the contest-success function, and show how different types of heterogeneity determine the presence of discouragement or encouragement effects.

### 3 Model

Consider a contest between two risk-neutral players  $i \in \{1, 2\}$  who compete for a single prize of value  $V > 0$ . Both players simultaneously choose effort  $e_i \geq 0$ , and the cost of effort  $c(e_i)$  is described by a continuously differentiable, strictly increasing and strictly convex function satisfying  $c(0) = 0$ . The skill (type) of player  $i$  is denoted by  $\Theta_i$ . There is uncertainty about skills, which means that  $\Theta_i$  is a random variable. The realization of  $\Theta_i$  is denoted by  $\theta_i$  and it is not known to any of the players (not even player  $i$ ). It is commonly known, however, that  $\Theta_i$  is independently and absolutely continuously distributed according to the pdf  $f_i$  (with cdf  $F_i$ ) with finite mean  $\mu_i$ . For a given density  $f$ , we will use  $\text{supp}(f) = \{x \in \mathbb{R} : f(x) > 0\}$  to denote its support. We assume that the supports of  $f_1$  and  $f_2$  overlap on a subset of  $\mathbb{R}$  with positive measure. Notice that standard

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<sup>6</sup>Some exceptions to the standard results in the context of the Tullock contest and the Lazear-Rosen tournament have already been documented in the literature. See, e.g., the work by Drugov and Ryvkin (2017) and Fu and Wu (2020) on biases in contests, and Lu, Wang, and Zhou (2021) on identity-dependent prizes.

continuous distributions can be employed with both bounded and unbounded supports. Moreover, the distributions can be different for the two players.

Symmetric uncertainty regarding skills is typically imposed in the career-concerns literature (e.g., Holmström 1982, Holmström and Ricard I Costa 1986, Dewatripont, Jewitt, and Tirole 1999, Auriol, Friebe, and Pechlivanos 2002, and Bar-Isaac and Lévy 2022) and also in the literature on promotion signaling (e.g., Waldman 1984, Bernhardt 1995, Owan 2004, Ghosh and Waldman 2010, DeVaro and Waldman 2012, and Gürtler and Gürtler 2019). This literature refers to firm-worker relationships, and the idea is that both firms and workers are uncertain about how well workers perform when they begin their working careers and that this uncertainty is reduced over time once performance information becomes available. We adopt this idea, referring to  $\Theta_i$  as a player's skill, but it is also possible to interpret it as noise, as it is often done in the contest literature.<sup>7</sup>

The production of player  $i$ , and hence his or her *contribution* to the contest, is given by the continuously differentiable production function  $g(\theta_i, e_i)$ .<sup>8</sup> Importantly, we assume that  $\frac{\partial g}{\partial \theta_i} > 0$  for all  $e_i > 0$  which means (realistically) that each player's contribution to the contest is increasing with respect to his or her skill, for a given level of effort. Player  $i$  wins the contest against the opponent player  $k \in \{1, 2\}$ ,  $k \neq i$ , if and only if the contribution of player  $i$  is strictly higher than the contribution of player  $k$ , namely,  $g(\theta_i, e_i) > g(\theta_k, e_k)$ .<sup>9</sup> We denote by  $P_i(e_i, e_k)$  player  $i$ 's probability of winning the contest (as a function of the efforts of both players) and we define the expected payoff as  $\pi_i(e_i, e_k) := P_i(e_i, e_k)V - c(e_i)$ . We also define  $\hat{e} := c^{-1}(V)$  and  $E := [0, \hat{e}]$ . A player's equilibrium effort will always belong to the set  $E$  as the probability of winning is bounded above by unity. We impose the following assumption:

**Assumption 1.** *The primitives of the model are such that: (i)  $\pi_i(e_i, e_k)$  is continuously differentiable, and, (ii) any interior solution of the system of first-order conditions for the players' problems of maximizing  $\pi_i(e_i, e_k)$  characterizes a pure-strategy Nash equilibrium.*

The validity of the first-order approach is typically ensured by imposing assump-

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<sup>7</sup>Many other interpretations are feasible as well. For instance,  $\Theta_i$  could simply account for player  $i$ 's luck in the contest. When the contest outcome is based on performance evaluation, it could capture measurement error. It could also represent bias or preferences of the supervisor, unknown to the players.

<sup>8</sup>In the literature on contests, a player's contribution to the contest is also denoted as a player's *score*.

<sup>9</sup>Note that, as usual in such contest models, adding a common random shock to the players' outputs does not affect the event of winning and, therefore, does not have an effect on the equilibrium. Also notice that  $g(\theta_i, e_i) = g(\theta_k, e_k)$  happens with probability zero. In the following, whenever we refer to two players  $i$  and  $k$ , we (implicitly) assume that  $i, k \in \{1, 2\}$ ,  $i \neq k$ .

tions on the primitives of the model that guarantee that the objective functions  $\pi_i$  are quasi-concave and increasing at  $e_i = 0$ . Previous papers in the contest-theory literature, however, have shown that the first-order approach may be valid even when the objective functions are neither quasi-concave nor increasing at  $e_i = 0$  (see, e.g., Figure 1 in Schweinzer and Segev 2012). As we do not want to rule out such cases, we assume that the Nash-equilibrium efforts are characterized by the players' first-order conditions to their maximization problems without restricting the shape of  $\pi_i$  too much.

Each of the theoretical results we present will be accompanied by at least one example for which we verify that the first-order conditions indeed characterize an equilibrium, by verifying the appropriate second-order conditions. As we permit a wide range of production functions and skill distributions, resulting in payoff functions that are not generally well-behaved, it is not feasible to pin down the exact set of parameters for which Assumption 1 is satisfied. However, in some instances, it is easy to verify that the first-order approach is valid, e.g., in the case of additive production functions and sufficiently convex cost functions. Moreover, given that the Lazear-Rosen tournament and the Tullock contest are special cases of our model, all that is known about equilibrium existence for these two models continues to hold in our setting.<sup>10</sup>

Finally, we assume that there exist  $\bar{e}_i, \check{e}_i \in \text{int } E$  such that  $\frac{\partial \pi_i(e_i, e_k)}{\partial e_i} |_{e_i=e_k=\bar{e}_i} < 0$  and  $\frac{\partial \pi_i(e_i, e_k)}{\partial e_i} |_{e_i=e_k=\check{e}_i} > 0$ . This ensures that the first-order condition to player  $i$ 's maximization problem can be fulfilled in a symmetric equilibrium.

## 4 Equilibrium characterization

We focus on pure-strategy Nash equilibria in which both players choose the same level of effort. The following lemma provides a sufficient condition for such a symmetric equilibrium to exist.

**Lemma 1.** *A sufficient condition for a symmetric equilibrium to exist is that  $\frac{\partial P_i(e_i, e_k)}{\partial e_i} |_{e_i=e_k=e}$  is the same for  $i, k \in \{1, 2\}, i \neq k$ , and all  $e \in \text{int } E$ .*

*Proof.* See Appendix A.1. □

<sup>10</sup>The Lazear-Rosen tournament is obtained in our model by setting  $g(\theta_i, e_i) = \theta_i + e_i$ . To see that the Tullock contest is a special case of our model, see Jia (2008) who demonstrates that a contest with a multiplicative production technology  $g(\theta_i, e_i) = \theta_i e_i$ , where  $\theta_i$  follows a certain class of distributions, results in a winning probability of the Tullock form. See also Clark and Riis (1996), Fullerton and McAfee (1999), and Fu and Lu (2012). The literature contains a range of modifications and generalizations of the Tullock contest-success function, some of which cannot be micro-founded in a similar way. See the recent discussion in Kirkegaard (2022).

We will make use of Lemma 1 to prove the existence of a symmetric equilibrium by checking the sufficient condition. Since this condition depends on the winning probability, we need to specify this probability first. For each  $e > 0$ , we define the function  $g_e : \mathbb{R} \rightarrow \mathbb{R}$  by  $g_e(x) = g(x, e)$ . The function  $g_e(x)$  is strictly increasing in  $x$  and thus invertible, and we denote the (strictly increasing) inverse by  $g_e^{-1}$ . This notation can be motivated by the fact that the event of player  $i$  winning over player  $k$  can be written as

$$\begin{aligned} g(\theta_k, e_k) &< g(\theta_i, e_i) \\ \Leftrightarrow g_{e_k}(\theta_k) &< g_{e_i}(\theta_i) \\ \Leftrightarrow \theta_k &< g_{e_k}^{-1}(g_{e_i}(\theta_i)). \end{aligned}$$

Considering all potential realizations of  $\Theta_i$  and  $\Theta_k$ , the winning probability of player  $i$  is

$$P_i(e_i, e_k) = \int_{\mathbb{R}} F_k(g_{e_k}^{-1}(g_{e_i}(x))) f_i(x) dx.$$

By symmetry, the winning probability of player  $k$  is

$$P_k(e_i, e_k) = \int_{\mathbb{R}} F_i(g_{e_i}^{-1}(g_{e_k}(x))) f_k(x) dx.$$

The derivative of player  $i$ 's winning probability with respect to  $e_i$  is given by:<sup>11</sup>

$$\frac{\partial P_i(e_i, e_k)}{\partial e_i} = \int_{\mathbb{R}} f_k(g_{e_k}^{-1}(g_{e_i}(x))) \frac{d}{de_i} (g_{e_k}^{-1}(g_{e_i}(x))) f_i(x) dx. \quad (1)$$

The derivative of player  $k$ 's winning probability with respect to  $e_k$  is given by:

$$\frac{\partial P_k(e_i, e_k)}{\partial e_k} = \int_{\mathbb{R}} f_i(g_{e_i}^{-1}(g_{e_k}(x))) \frac{d}{de_k} (g_{e_i}^{-1}(g_{e_k}(x))) f_k(x) dx. \quad (2)$$

It can immediately be seen that expressions (1) and (2) are equal when  $e_i = e_k = e \in \text{int } E$  since, in this case,  $g_{e_k}^{-1}(g_{e_i}(x)) = g_{e_i}^{-1}(g_{e_k}(x)) = x$  and  $\frac{d}{de_k} g_{e_i}^{-1}(g_{e_k}(x)) = \frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(x))$ . Thus, the sufficient condition for the existence of a symmetric equilibrium in Lemma 1 is satisfied. Hence, we have the following theorem.

**Theorem 1.** *There exists a symmetric equilibrium in which both players choose the same level of effort.*

*Proof.* See Appendix A.2. □

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<sup>11</sup>Notice that  $F_k$  is differentiable almost everywhere, since it is the cdf of the absolutely continuous random variable  $\Theta_k$  with  $f_k$  as the corresponding pdf.



The theorem states that, even if the players are asymmetric (i.e.,  $f_1 \neq f_2$ ), there always exists a symmetric equilibrium of the contest game. This key result allows a tractable analysis of contests between asymmetric players in a variety of different settings. We define  $a_e : \mathbb{R} \rightarrow \mathbb{R}$  by

$$a_e(x) = \frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(x)) \Big|_{e_i=e_k=e} = \frac{\partial g(x, e)}{\partial e} / \frac{\partial g(x, e)}{\partial x} = MRTS(x, e), \quad (3)$$

where the equality follows from an application of the inverse function theorem and  $MRTS(x, e)$  denotes the marginal rate of technical substitution between skill and effort in a symmetric equilibrium.<sup>12</sup> Recognizing that the two players have the same cost function  $c(e)$ , we can write the (identical) first-order condition for effort for the two players in a symmetric equilibrium as

$$V \int_{\mathbb{R}} a_{e^*}(x) f_k(x) f_i(x) dx = c'(e^*). \quad (4)$$

The LHS of (4) describes the return to a marginal increase in effort, averaged over all possible skill realizations. The key observation necessary to understand the intuition behind (4) is that a marginal effort increase is pivotal to winning the contest if and only if  $g(\theta_k, e_k) = g(\theta_i, e_i)$ . In a symmetric equilibrium, in which  $e_k = e_i$ , this implies that  $\theta_k = \theta_i$ . Accordingly, equation (4) contains the “collision density”  $f_k(x)f_i(x)$  that describes how likely it is that the skill realizations of the two competing players are the same. The fact that this term is the same for both players is due to our assumption of symmetric uncertainty. Furthermore, in situations in which a marginal effort increase is pivotal to winning the contest,  $a_e(x)$  is the same for both players. This is because  $g(\theta, e)$  is the same for both players, and depends only on effort  $e$  and skill  $\theta$ .

The function  $a_e(x)$  describes how a marginal increase in effort by a player increases output relative to his or her rival and is equal to the marginal rate of technical substitution between skill and effort. The purpose of raising effort is to beat players with higher skill. The MRTS determines the range of additional types that the player can win against through a small effort increase. The lower is the sensitivity of output to skill in the production function, the smaller is the advantage of marginally more skilled rivals, and the higher is the marginal incentive to exert effort. A direct implication is that the marginal incentive to exert effort is higher in environments in which players’ outputs depend to a large degree on effort than in those in which the output is mainly determined

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<sup>12</sup>To see this, notice that  $\frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(x)) \Big|_{e_i=e_k=e} = \frac{1}{g'_{e_k}(g_{e_k}^{-1}(g_{e_i}(x)))} \frac{d}{de_i} g_{e_i}(x) \Big|_{e_i=e_k=e} = \frac{\frac{d}{de} g_e(x)}{g'_e(x)}$ .

by players' skills. The reason is that the MRTS tends to be larger in the former than in the latter environments, implying a greater impact of effort on the winning probability, as just explained.

## 4.1 Limitations and Generalizations

Before proceeding, we would like to discuss certain limitations and generalizations of our findings. The first limitation is that a symmetric equilibrium generally does not exist when players are privately informed about their skills. The reason is simple. We have shown before that a symmetric equilibrium exists since the collision density  $f_k(x)f_i(x)$  is the same from both players' point of view. In contrast, when players are privately informed about their own skills, they are only uncertain about the skill of their opponent. This means that the collision density is replaced by  $f_k(x)$  for player  $i$  and  $f_i(x)$  for player  $k$ , introducing an asymmetry into the model.<sup>13</sup>

The second limitation is that the general result about the existence of a symmetric equilibrium does not extend to the  $n$ -player case.<sup>14</sup> There is again a simple explanation for this. A player will only win the contest if he or she beats all other players. Essentially, each player is thus competing against the best of the other players, that is, the highest order statistic, and the collision density is generalized accordingly (as the product of the player's own skill density and the density of the highest order statistic of the other players). This implies that the collision density is no longer the same from all players' point of view.<sup>15</sup>

Regarding generalizations, while we have shown that a symmetric equilibrium generally exists if the skill distribution is the source of the heterogeneity, this is no longer the case if the contestants are asymmetric regarding prize valuations, cost functions or production functions. Nevertheless, under stricter assumptions, we can identify some setups in which the analysis of the baseline case can be generalized. This analysis is presented in Appendix A.8.

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<sup>13</sup>The formal details are presented in Appendix A.5.

<sup>14</sup>Contests with more than two players have been studied by, e.g., Tullock (1980), Nalebuff and Stiglitz (1983), Hillman and Riley (1989), Zájbojník and Bernhardt (2001), Chen (2003), Zájbojník (2012), and Ryvkin and Drugov (2020).

<sup>15</sup>The formal details are presented in Appendix A.6. We have identified one situation where a symmetric equilibrium exists in the  $n$ -player case which pertains to the situation where  $n - 1$  equally skilled players compete against a player who is more highly skilled (as in Brown 2011 and Krumer, Megidish, and Sela 2017), and the skill distribution is given by the reflected exponential distribution. The analysis of this case is presented in Appendix A.7.

## 5 Comparative statics results

In this section, we investigate how heterogeneity in the statistical properties of the skill distributions of the competing players influences the incentive to exert effort. To facilitate the derivation of these results, we define  $r_{e,i} : \mathbb{R} \rightarrow \mathbb{R}$  given by  $r_{e,i}(x) = a_e(x)f_i(x)$ . Equation (4) can thus be written as:

$$V \int_{\mathbb{R}} r_{e^*,i}(x)f_k(x)dx = c'(e^*). \quad (5)$$

The integral now has the same structure as a decision maker's expected utility in decision theory (e.g., Levy 1992), where the function  $r_{e,i}$  corresponds to the decision maker's utility function. As we will see, this link proves useful in deriving several key results. We also need one additional assumption:

**Assumption 2.** *The primitives of the model are such that  $q : E \rightarrow \mathbb{R}$ , defined by*

$$q(e) = V \int_{\mathbb{R}} r_{e,i}(x)f_k(x)dx - c'(e),$$

*is strictly decreasing.*

As  $c$  is strictly convex, Assumption 2 is not very strong and is always satisfied if  $\int_{\mathbb{R}} r_{e,i}(x)f_k(x)dx$  is non-increasing in  $e$ . To give a specific example, consider the CES production function  $g(\theta_i, e_i) = (\alpha\theta_i^\rho + \beta e_i^\rho)^{\frac{1}{\rho}}$ , with  $\alpha, \beta > 0$  and  $\rho \leq 1$ . Here  $a_e(x) = \frac{\beta}{\alpha} \left(\frac{x}{e}\right)^{1-\rho}$ , implying that  $\int_{\mathbb{R}} a_e(x)f_1(x)f_2(x)dx = e^{\rho-1} \int_{\mathbb{R}} \frac{\beta}{\alpha} x^{1-\rho} f_1(x)f_2(x)dx$ . For this specification, Assumption 2 is satisfied in all cases where players have an incentive to exert positive effort (i.e.,  $\int_{\mathbb{R}} \frac{\beta}{\alpha} x^{1-\rho} f_1(x)f_2(x)dx > 0$ ). Furthermore, the assumption ensures that effort is always increasing in the prize and that the considered equilibrium is unique in the class of symmetric equilibria (the latter result follows from the assumption ensuring that there is a unique  $e$  solving equation (5)).

### 5.1 First-order stochastic dominance

A standard result in contest theory is that heterogeneity among players with respect to their skills reduces the incentive to exert effort (see, e.g., Schotter and Weigelt 1992 or Observation 1 in the survey by Chowdhury, Esteve-Gonzalez, and Mukherjee 2019). In our model, this standard result is potentially reversed, as we will now show.

Consider a contest with two players with skills drawn from two distributions with expected values  $\mu_k$  and  $\mu_i$ , respectively. If, from the outset,  $\mu_k \geq \mu_i$  and the difference

$\mu_k - \mu_i$  is increased, then the two players become more heterogeneous in terms of their expected skill. Based on this idea, we proceed by investigating the consequences of making players more heterogeneous in the sense of first-order stochastic dominance, as captured by the following definition.

**Definition 1.** Let  $\mu_k$  and  $\mu_i$  refer to the expected values of the skill distributions  $(F_k, F_i)$  in an initial contest. Players in a contest with skill distributions  $(\tilde{F}_k, F_i)$  are said to be more heterogeneous (with respect to their skills) relative to players in the initial contest with skill distributions  $(F_k, F_i)$ , in a first-order sense, if either of the following conditions hold:

- (i)  $\mu_k \geq \mu_i$  and  $\tilde{F}_k$  dominates  $F_k$  in the sense of first-order stochastic dominance.
- (ii)  $\mu_k \leq \mu_i$  and  $\tilde{F}_k$  is dominated by  $F_k$  in the sense of first-order stochastic dominance.

Due to Assumption 2, equilibrium effort increases if a change in the primitives of the model leads to an increase in  $\int_{\mathbb{R}} r_{e,i}(x) f_k(x) dx$ . As indicated before, this expression has the same structure as a decision maker's expected utility in decision theory, where the function  $r_{e,i}$  is replaced by the decision maker's utility function. Since the structure of the problems is the same, we can make extensive use of results from decision theory in our analysis. We obtain the following proposition.

**Proposition 1.** Consider two contests with skill distributions  $(\tilde{F}_k, F_i)$  and  $(F_k, F_i)$  where  $\text{supp}(\tilde{f}_k)$  and  $\text{supp}(f_k)$  both are subsets of  $\text{supp}(f_i)$ . Let  $\tilde{e}^*$  and  $e^*$  denote, respectively, the (symmetric) equilibrium efforts associated with these contests. Then,  $\tilde{e}^* > e^*$  if either one of the following statements hold:

- (i)  $r_{e,i}(x)$  is strictly increasing for all  $x \in \text{supp}(f_i)$  and all  $e \geq 0$ , and  $\tilde{F}_k$  dominates  $F_k$  in the sense of first-order stochastic dominance.
- (ii)  $r_{e,i}(x)$  is strictly decreasing for all  $x \in \text{supp}(f_i)$  and all  $e \geq 0$ , and  $\tilde{F}_k$  is dominated by  $F_k$  in the sense of first-order stochastic dominance.

*Proof.* See Appendix A.3. □

Note that Proposition 1 holds independently of whether  $\mu_k \leq \mu_i$  or  $\mu_k \geq \mu_i$ . Combining Definition 1 with Proposition 1, we have the following corollary.<sup>16</sup>

<sup>16</sup>There is one small caveat to Corollary 1 that we should mention. If equilibrium effort increases as contestants become more heterogeneous, then a symmetric equilibrium in which both players exert positive effort will fail to exist if the heterogeneity between players becomes too large. The reason is that the weaker player would eventually receive a negative payoff, meaning that this player would prefer to choose zero effort.

**Corollary 1.** *Effort can be higher when contestants are more heterogeneous in a first-order sense.*

We illustrate the intuition behind Proposition 1 and Corollary 1 through two examples. In each example, we start from a situation of equal expected skills, and then introduce a first-order stochastic dominance shift. In the first example, which has a somewhat simpler intuition than the second,  $r_{e,i}(x)$  is strictly decreasing and effort gets higher as player  $k$  becomes weaker, illustrating part (ii) of Proposition 1. In the second example,  $r_{e,i}(x)$  is strictly increasing and effort gets higher as player  $k$  becomes stronger, illustrating part (i) of Proposition 1.

**Example 1.** *Suppose that  $g(\theta, e) = \theta + e$ ,  $\Theta_i \sim \text{Exp}(\frac{4}{3})$ ,  $\Theta_k \sim U[\frac{1}{2}, 1]$ ,  $\tilde{\Theta}_k \sim U[\frac{7}{16}, \frac{15}{16}]$ ,  $c(e) = \frac{e^2}{2}$ ,  $V = 1$ . Then  $e^* = \frac{2(\exp(\frac{2}{3})-1)}{\exp(\frac{4}{3})} \approx 0.499$  and  $\tilde{e}^* = \frac{2(\exp(\frac{2}{3})-1)}{\exp(\frac{5}{4})} \approx 0.543$ .*

In Example 1, the first thing to notice is that the additive production technology implies that  $a_e(x) = 1$ . This further implies that  $r_{e,i}(x)$  is strictly decreasing for all relevant  $x$ , since  $f_i(x)$  is the decreasing pdf of the exponential skill distribution. The fact that  $a_e(x) = 1$  also implies that the incentive to supply effort, as given by (4), only depends on the collision density  $f_k(x)f_i(x)$ . Since  $f_i(x)$  is decreasing, and  $f_k(x)$  is uniform and shifted to the left, the collision density between  $\tilde{f}_k$  and  $f_i$  is everywhere larger than the collision density between  $f_k$  and  $f_i$ , see Figure 1 for an illustration. Recalling the discussion following (4), the considered distributional shift makes the pivotal situations more “likely”. Thus, both players have a higher incentive to exert effort.

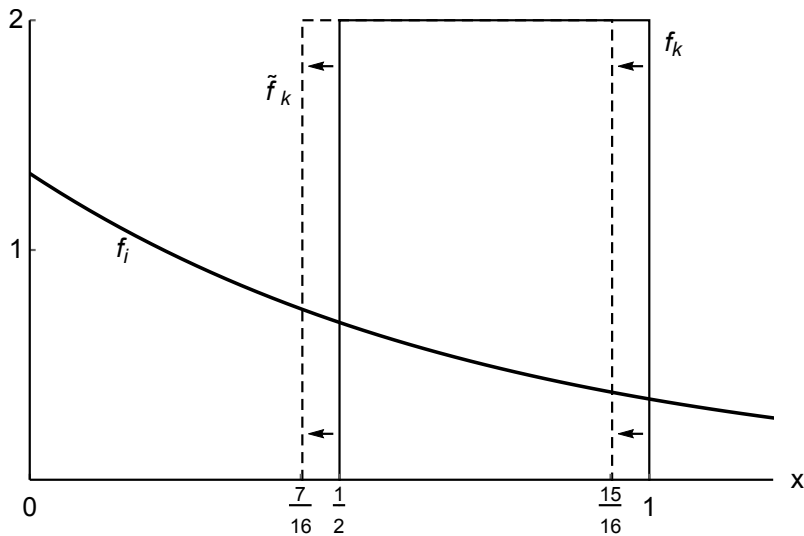


Figure 1: Illustration of Example 1

**Example 2.** Suppose that  $g(\theta, e) = \theta \cdot e$ ,  $\Theta_i \sim U[0, 1]$ ,  $\Theta_k \sim U[\frac{1}{4}, \frac{3}{4}]$ ,  $\tilde{\Theta}_k \sim U[\frac{5}{16}, \frac{13}{16}]$ ,  $c(e) = \frac{e^2}{2}$ ,  $V = 1$ . Then  $e^* = \frac{1}{\sqrt{2}} \approx 0.707$  and  $\tilde{e}^* = \frac{3}{4} = 0.75$ .

In Example 2, the multiplicative production technology implies that  $a_e(x) = x/e$  which is a strictly increasing function of  $x$ . This further implies that  $r_{e,i}(x)$  is strictly increasing on  $[0, 1]$  because  $f_i$  is uniform. The shift in the skill distribution of player  $k$  from  $F_k$  to  $\tilde{F}_k$  implies that the expected skill of player  $k$  increases. However, the height of the density of player  $k$ 's skill distribution does not change ( $f_k(x) = 2$ ,  $x \in [\frac{1}{4}, \frac{3}{4}]$  and  $\tilde{f}_k(x) = 2$ ,  $x \in [\frac{5}{16}, \frac{13}{16}]$ ). Thus, since  $f_i(x) = 1$ , we have that  $f_k(x)f_i(x) = \tilde{f}_k(x)f_i(x) = 2$  at all points where these collision densities are non-zero. However, due to the distributional shift, the subset of  $\mathbb{R}$  where the two uniform distributions overlap shifts to the right. Therefore, the two distributions collide at larger values of  $x$  (see Figure 2 for an illustration). This would have no effect on the incentive to exert effort if  $a_e(x)$  were constant, as in Example 1. However, in the current example, we have that  $a_e(x) = x/e$ . Thus, taking into account the three terms in (4), the fact that the two distributions collide at larger values of  $x$  increases the incentive to exert effort for both players. Intuitively, given that the pivotal situations now occur at larger values of skill, the fact that there is a complementarity between skill and effort in the production function implies that the incentive to supply effort is higher for both players.

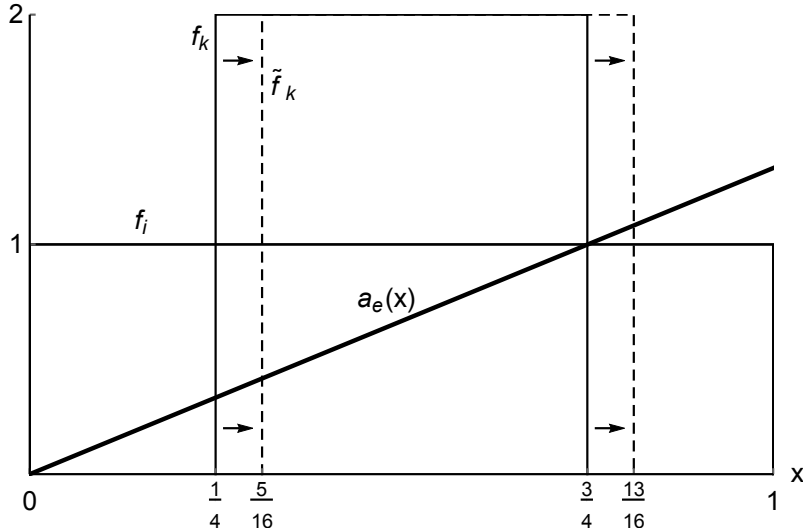


Figure 2: Illustration of Example 2

Concluding this section, we note that the conditions in Proposition 1 are sufficient, but not necessary for the result that effort can be higher when contestants are more heterogeneous. To illustrate this, we present an illustrative example based on normal

distributions where we determine the marginal winning probability in a situation with symmetric effort.

**Example 3.** Suppose that  $\Theta_i \sim N(\mu_i, \sigma_i^2)$ ,  $\Theta_k \sim N(\mu_k, \sigma_k^2)$ , and  $g(\theta, e) = \theta \cdot e$  and let  $(\sigma_i, \sigma_k) = (1, 1)$ ,  $(\mu_i, \mu_k) = (\frac{1}{2}, \frac{1}{2})$ ,  $V = 1$ , and  $c(e) = \frac{e^2}{2}$ . Then equilibrium effort is  $e^* = \left(2\pi^{\frac{1}{4}}\right)^{-1} \approx 0.38$ . If we increase  $\mu_i$  from  $\frac{1}{2}$  to  $\frac{3}{2}$ , keeping  $\mu_k$  constant, equilibrium effort increases to  $\tilde{e}^* = \left(\sqrt{2} \exp\left(\frac{1}{8}\right) \pi^{\frac{1}{4}}\right)^{-1} \approx 0.47$ .

In the above example it can be verified that  $r_{e,i}(x) = a_e(x)f_i(x)$  is neither always increasing nor always decreasing, by virtue of the multiplicative production technology combined with the bell-shaped normal distribution. Nonetheless, equilibrium effort increases as players become more heterogeneous in the sense of increasing the distance  $|\mu_i - \mu_k|$ .

## 5.2 Mean-preserving spreads

The studies by Hvide (2002), Kräkel and Sliwka (2004), Kräkel (2008), Gilpatric (2009), and DeVaro and Kauhanen (2016) investigate how “risk” or “uncertainty” affects players’ incentive to exert effort in contests. One result that is common to all of these analyses is that in contests between equally able players, higher risk (as measured by a higher variance of the random variables capturing the uncertainty of the contest outcome) leads to lower efforts. We revisit this result in the context of our model, and show that effort may increase as the skill distribution of one of the players becomes more uncertain.

The economic literature has identified different ways to conceptualize risk or uncertainty. We follow Rothschild and Stiglitz 1970 by using the concept of a mean-preserving spread to measure the uncertainty regarding players’ skill distributions.<sup>17</sup>

**Definition 2.** The skill distribution  $\tilde{F}_i$  is said to be more uncertain than the distribution  $F_i$  if  $\tilde{F}_i$  is a mean-preserving spread of  $F_i$ .

Equipped with this definition, we can use well-known results from decision theory to obtain our next proposition:

**Proposition 2.** Consider two contests with skill distributions  $(\tilde{F}_k, F_i)$  and  $(F_k, F_i)$  where  $\text{supp}(\tilde{f}_k)$  and  $\text{supp}(f_k)$  both are subsets of  $\text{supp}(f_i)$ . Let  $\tilde{e}^*$  and  $e^*$  denote, respectively, the

<sup>17</sup>Gerchak and He (2003) analyze how effort in two-player contests is determined by the Rényi entropy, and Drugov and Ryvkin (2020) generalize their insights to the case of more than two players. Their results, however, rely on the assumptions of an additive production function and homogeneous players (i.e., players with the same skill distributions). It is not obvious how these results transfer to a model with general production technologies and asymmetric skill distributions, which is the primary focus here.

(symmetric) equilibrium efforts associated with these contests. Suppose that  $\tilde{F}_k$  is more uncertain than  $F_k$ . Then, the following results hold:

- (i) If  $r_{e,i}(x)$  is strictly convex on  $\text{supp}(f_i)$  for all  $e \geq 0$ , then  $\tilde{e}^* > e^*$ .
- (ii) If  $r_{e,i}(x)$  is linear on  $\text{supp}(f_i)$  for all  $e \geq 0$ , then  $\tilde{e}^* = e^*$ .
- (iii) If  $r_{e,i}(x)$  is strictly concave on  $\text{supp}(f_i)$  for all  $e \geq 0$ , then  $\tilde{e}^* < e^*$ .

*Proof.* See Appendix A.4. □

The key insight needed to understand Proposition 2 is that applying a mean-preserving spread to the distribution  $F_k$  shifts probability mass from the center to the tails of the distribution, and the impact of this change on the incentive to exert effort depends on the curvature of  $r_{e,i}(x)$ . Notice that Proposition 2 also holds if players have the same expected skill, namely  $\mu_i = \mu_k$ . This means that, in a contest with two players who are expected to be equally able, higher uncertainty regarding players' skills may increase the incentive to exert effort.

Next, we illustrate and provide intuition for Proposition 2 by presenting an example set in the context of the Lazear-Rosen model with an additive production technology. The example demonstrates that increasing the uncertainty of the contest while keeping the expected skill of both players unchanged, can increase equilibrium effort.

**Example 4.** Consider a contest with the additive production function  $g(\theta, e) = \theta + e$ , the parameter  $V = 1$ , and the cost function  $c(e) = \frac{e^2}{2}$ . Suppose  $\Theta_i \sim \text{Exp}(1)$  and  $\Theta_k \sim U\left[\frac{1}{2}, \frac{3}{2}\right]$  (implying  $\mu_i = \mu_k = 1$ ). Equilibrium effort is then  $e^* = \frac{\exp(1)-1}{\exp(\frac{3}{2})} \approx 0.38$ . Now, consider a mean-preserving spread of the skill distribution of player  $k$ , enlarging the support of the uniform distribution, such that  $\tilde{\Theta}_k \sim U[0, 2]$ . Then effort increases to  $\tilde{e}^* = \frac{\exp(2)-1}{2\exp(2)} \approx 0.43$ .

In Example 4, we have imposed the additive production technology which implies  $a_e(x) = 1$ . Thus, the convexity of  $r_{e,i}(x)$  referred to in part (i) of Proposition 2 is determined by the convexity of  $f_i(x)$ . To understand how the shift from  $f_k$  to  $\tilde{f}_k$  affects the incentive to exert effort, we need to study how the integral in (4) is affected. Similar to Example 1, given that  $a_e(x) = 1$ , it is sufficient to compare  $\int f_i(x)f_k(x)dx$  with  $\int f_i(x)\tilde{f}_k(x)dx$ . The shift from  $f_k$  to  $\tilde{f}_k$  entails an enlargement of the support of the uniform distribution. This implies that the density decreases for intermediate values of  $x$ , but increases for low and high values of  $x$  (see Figure 3 for an illustration). Given that  $f_i(x)$  is strictly decreasing, the part of the skill distribution of player  $k$  that is stretched



out to the left will collide with relatively large values of  $f_i$ , whereas the the part of the skill distribution of player  $k$  that is stretched out to the right will collide with relatively small values of  $f_i$ , creating a trade-off. The fact that  $f_i$  is not only strictly decreasing, but also convex, resolves this trade-off, implying that the overall effect of the shift is to increase the value of the integral expression. Thus, both players have a higher incentive to exert effort as a result of the move from  $f_k$  to  $\tilde{f}_k$ .

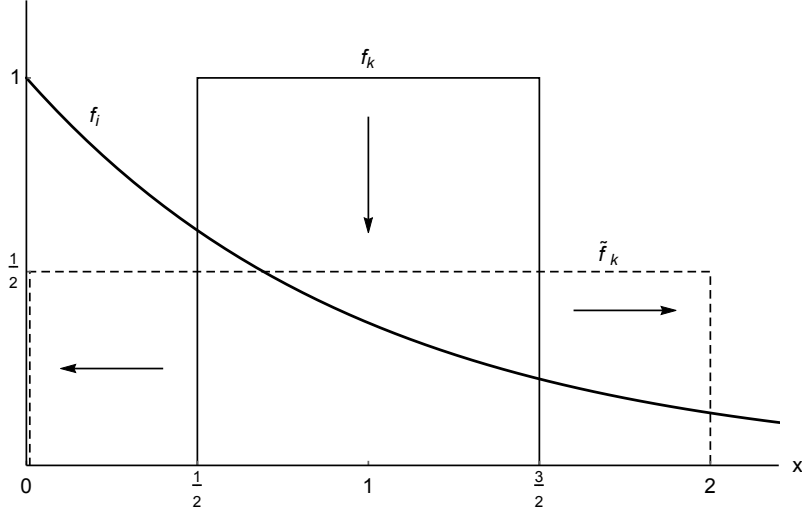


Figure 3: Illustration of Example 4

We conclude this section by defining contestant heterogeneity in a *second-order* sense and we follow the structure of the corresponding definition of heterogeneity in a first-order sense (Definition 1).

**Definition 3.** *Players in a contest with skill distributions  $(\tilde{F}_k, F_i)$ , are said to be more heterogeneous (with respect to their skills) relative to players in the initial contest with skill distributions  $(F_k, F_i)$ , in a second-order sense, if either of the following conditions hold:*

- (i)  $\tilde{F}_k$  is a mean-preserving spread over  $F_k$ , which in turn is a mean-preserving spread over  $F_i$ .
- (ii)  $F_i$  is a mean-preserving spread over  $F_k$ , which in turn is a mean-preserving spread over  $\tilde{F}_k$ .

Combining Proposition 2 with Definition 3, we have the following corollary.

**Corollary 2.** *Effort can be higher when contestants are more heterogeneous in a second-order sense.*

## **6 Concluding remarks**

We have explored simple equilibria in contests between two heterogeneous contestants. Under general assumptions about the production technology and the skill distributions of the players, we have shown that the contest has a symmetric equilibrium in which both players exert the same effort. We have also provided intuition regarding how the different components of the contest interact to determine the incentive to exert effort and revisited several comparative statics results of contest theory, showing that standard results in the literature are not necessarily robust to generalizations of the production technology or skill distributions. In particular, we have found that making players more heterogeneous can increase the incentive to exert effort.

# A Appendix

## A.1 Proof of Lemma 1

Suppose that  $\frac{\partial P_i(e_i, e_k)}{\partial e_i}|_{e_i=e_k=e}$  is the same for both  $i \in \{1, 2\}$  and all  $e \in \text{int } E$ . Then we have  $\frac{\partial P_1(e_1, e_2)}{\partial e_1}|_{e_1=e_2=e} V - c'(e) = \frac{\partial P_2(e_2, e_1)}{\partial e_2}|_{e_1=e_2=e} V - c'(e)$  for all  $e \in \text{int } E$ . Since  $\pi_i(e_i, e_k)$  is continuously differentiable,  $\frac{\partial P_i(e_i, e_k)}{\partial e_i}|_{e_i=e_k=e} V - c'(e)$  is a continuous function of  $e$ . Furthermore, recall that there exist  $\bar{e}_i, \check{e}_i \in \text{int } E$  such that  $\frac{\partial \pi_i(e_i, e_k)}{\partial e_i}|_{e_i=e_k=\bar{e}_i} < 0$  and  $\frac{\partial \pi_i(e_i, e_k)}{\partial e_i}|_{e_i=e_k=\check{e}_i} > 0$ . Hence, by the Intermediate Value Theorem, there is some  $e^* \in \text{int } E$  such that  $\frac{\partial P_i(e_i, e_k)}{\partial e_i}|_{e_i=e_k=e^*} V - c'(e^*) = 0$ . By Assumption 1,  $e_1 = e_2 = e^*$  is a Nash equilibrium.

## A.2 Proof of Theorem 1

Since we wish to apply the sufficient condition from Lemma 1, we restrict attention to  $e_i > 0$ . Then, the function  $g_e : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $g_e(x) = g(x, e)$  is strictly increasing and, thus, invertible. The inverse,  $g_e^{-1}$ , is strictly increasing as well. For the two (different) players  $i, k \in \{1, 2\}$ , we observe

$$\begin{aligned} g(\theta_i, e_i) &< g(\theta_k, e_k) \\ \Leftrightarrow g_{e_i}(\theta_i) &< g_{e_k}(\theta_k) \\ \Leftrightarrow \theta_i &< g_{e_i}^{-1}(g_{e_k}(\theta_k)). \end{aligned}$$

Player  $k$  thus wins with probability

$$\int F_i(g_{e_i}^{-1}(g_{e_k}(x))) f_k(x) dx.$$

Differentiating with respect to  $e_k$ , we obtain

$$\int f_i(g_{e_i}^{-1}(g_{e_k}(x))) \left( \frac{d}{de_k} g_{e_i}^{-1}(g_{e_k}(x)) \right) f_k(x) dx.$$

According to Lemma 1, and noting that  $g_{e_k}^{-1}(g_{e_i}(x)) = g_{e_i}^{-1}(g_{e_k}(x)) = x$  if  $e_i = e_k$ , a sufficient condition for a symmetric equilibrium to exist is that

$$\begin{aligned} &\int \left( \frac{d}{de_1} g_{e_2}^{-1}(g_{e_1}(x)) \Big|_{e_1=e_2=e} \right) f_1(x) f_2(x) dx \\ &= \int \left( \frac{d}{de_2} g_{e_1}^{-1}(g_{e_2}(x)) \Big|_{e_1=e_2=e} \right) f_1(x) f_2(x) dx, \end{aligned}$$

for all  $e \in \text{int } E$ . Since  $\frac{d}{de_1} g_{e_2}^{-1}(g_{e_1}(x)) \Big|_{e_1=e_2=e} = \frac{d}{de_2} g_{e_1}^{-1}(g_{e_2}(x)) \Big|_{e_1=e_2=e}$ , this condition is always fulfilled.

### A.3 Proof of Proposition 1

Suppose that Assumption 2 holds, and consider case (i), i.e.,  $r_{e,i}(x)$  is monotonically increasing in  $x$ , and  $\tilde{F}_k$  first-order stochastically dominates  $F_k$ . Denote the equilibrium effort levels for the two contests by  $\tilde{e}^*$  and  $e^*$ , respectively. Our goal is to show that  $\tilde{e}^* > e^*$ . The proof proceeds by contradiction, so suppose  $\tilde{e}^* \leq e^*$ . Observe that:

$$\begin{aligned} V \int r_{\tilde{e}^*,i}(x) \tilde{f}_k(x) dx - c'(\tilde{e}^*) &\geq \\ V \int r_{e^*,i}(x) \tilde{f}_k(x) dx - c'(e^*) &> \\ V \int r_{e^*,i}(x) f_k(x) dx - c'(e^*) &= 0. \end{aligned}$$

The first inequality follows from  $\tilde{e}^* \leq e^*$  together with Assumption 2. The second inequality follows from  $r_{e,i}(x)$  being monotonically increasing on  $\text{supp}(f_i)$ ,  $\tilde{F}_k$  first-order stochastically dominating  $F_k$ , and the fact that we have assumed that both  $\text{supp}(\tilde{f}_k)$  and  $\text{supp}(f_k)$  are subsets of  $\text{supp}(f_i)$ .<sup>18</sup> The equality follows since  $e^*$  is characterized by the first-order condition  $V \int r_{e^*,i}(x) f_k(x) dx - c'(e^*) = 0$ . We conclude that

$$V \int r_{\tilde{e}^*,i}(x) \tilde{f}_k(x) dx - c'(\tilde{e}^*) > 0.$$

This shows that the first-order condition for equilibrium effort cannot be fulfilled in the case of the distribution  $\tilde{F}_k$ , giving us the desired contradiction. By an analogous argument, we can show that  $\tilde{e}^* > e^*$  also in case (ii) where  $r_{e,i}$  is monotonically decreasing in  $x$  for all  $e > 0$  and  $F_k$  first-order stochastically dominates  $\tilde{F}_k$ . In this case,  $\int r_{e,i}(x) \tilde{f}_k(x) dx > \int r_{e,i}(x) f_k(x) dx$  for all  $e > 0$  (see, e.g., Levy 1992, p.557).

### A.4 Proof of Proposition 2

Because of Assumption 2 and the condition characterizing equilibrium effort, we need to show that  $\int r_{e,i}(x) \tilde{f}_k(x) dx > (=, <) \int r_{e,i}(x) f_k(x) dx$  if  $r_{e,i}$  is convex (linear, concave). The proof is very similar to part a) of the proof of Theorem 2 in Rothschild and Stiglitz (1970, p.237). In the case of convex  $r_{e,i}$ , the inequality in their proof is reversed, while it is

<sup>18</sup>See, e.g., Levy 1992, p.557. Notice that, in decision theory, the utility function is defined for all possible payoffs and therefore no additional constraints regarding the statistical supports need to be imposed.

replaced by an equality if  $r_{e,i}$  is linear.

## A.5 Privately known skills

Here we briefly comment on how our analysis is affected by assuming that players have private information regarding their own skill. The private-information assumption effectively implies that player  $i$  can, in a deterministic manner, choose output  $g(\theta_i, e_i)$  by making the appropriate effort choice  $e_i$ . The decision problem of player  $i$  can therefore, equivalently, be expressed as the specification of optimal effort  $e_i(\theta_i)$  or the choice of optimal output  $z_i(\theta_i) := g(\theta_i, e_i(\theta_i))$ , as a best response to the opponents' choice of effort or output. Assuming player  $i$  competes against another player  $k$ , player  $i$  wins the contest for given realizations of  $\Theta_i$  and  $\Theta_k$  if and only if the following condition holds

$$g(\theta_k, e_k) < g(\theta_i, e_i) \iff z_k(\theta_k) < z_i(\theta_i) \iff \theta_k < z_k^{-1}(z_i(\theta_i)),$$

where output is assumed to be strictly increasing in skill so that  $z_i$  is invertible with inverse  $z_i^{-1}$ . Taking into account that, from the perspective of player  $i$ , the uncertainty of the contest only concerns the skill realization of player  $k$ , we have that equilibrium efforts  $e_i(\theta_i)$  and  $e_k(\theta_k)$  satisfy:

$$\begin{aligned} e_i(\theta_i) &\in \operatorname{argmax}_{e_i} \left\{ F_k(z_k^{-1}(z_i(\theta_i)))V - c(e_i) \right\}, \\ e_k(\theta_k) &\in \operatorname{argmax}_{e_k} \left\{ F_i(z_i^{-1}(z_k(\theta_k)))V - c(e_k) \right\}. \end{aligned}$$

Thus, we can deduce that the first-order condition for player  $i$  only involves the skill distribution of the opposing player  $k$ , whereas the first-order condition for player  $k$  only involves the skill distribution of the opposing player  $i$ . Hence, the symmetry that was present in the main model, where the first-order condition for each player involved the product of  $f_i$  and  $f_k$  (see equation (4)), vanishes when skills are privately known, and the equilibrium effort functions  $e_i(\theta_i)$  and  $e_k(\theta_k)$  are no longer expected to be symmetric.<sup>19</sup>

## A.6 Lack of symmetry in the $n$ -player case

Consider the general case with  $n > 2$  players. Suppose for simplicity that  $g(\theta_i, e_i) = \theta_i + e_i$ , implying that  $a_e(x) = 1$ . The marginal winning probabilities for players  $i$  and  $k$

<sup>19</sup>The  $n$ -player case with privately known skills can be handled in an almost identical fashion. Instead of competing against player  $k$ , player  $i$  can be viewed as competing against the strongest of the opponents  $j \in \{1, \dots, n\}$ ,  $j \neq i$ , in the sense of the highest order statistic.

in a symmetric equilibrium ( $e_1 = \dots = e_n = e$ ) can be written as:

$$\begin{aligned}\frac{\partial P_i}{\partial e_i} &= \int_{\mathbb{R}} f_i(x) \frac{d}{dx} \left( F_k(x) \prod_{j \neq i, k} F_j(x) \right) dx \\ &= \int_{\mathbb{R}} \left( f_i(x) f_k(x) \prod_{j \neq i, k} F_j(x) + f_i(x) F_k(x) \frac{d}{dx} \left( \prod_{j \neq i, k} F_j(x) \right) \right) dx\end{aligned}\quad (\text{A.1})$$

and

$$\begin{aligned}\frac{\partial P_k}{\partial e_k} &= \int_{\mathbb{R}} f_k(x) \frac{d}{dx} \left( F_i(x) \prod_{j \neq i, k} F_j(x) \right) dx \\ &= \int_{\mathbb{R}} \left( f_k(x) f_i(x) \prod_{j \neq i, k} F_j(x) + f_k(x) F_i(x) \frac{d}{dx} \left( \prod_{j \neq i, k} F_j(x) \right) \right) dx.\end{aligned}\quad (\text{A.2})$$

The first term in (A.1) and (A.2) reflects the situation in which all players  $j \in \{1, \dots, n\}, j \neq i, k$  perform worse than players  $i$  and  $k$  so that the  $n$ -player contest collapses to a contest between players  $i$  and  $k$ . For this subcontest, the marginal winning probabilities are the same, as shown in the analysis of the two-player contest. The second term in (A.1) corresponds to the situation in which player  $i$  outperforms his or her rival  $k$ , such that the contest boils down to a contest between player  $i$  and the strongest of the players  $j \in \{1, \dots, n\}, j \neq i, k$ . The interpretation of the second term in (A.2) is analogous, with the role of  $i$  and  $k$  interchanged. Setting (A.1) equal to (A.2), we obtain

$$\begin{aligned}\int_{\mathbb{R}} f_i(x) F_k(x) \frac{d}{dx} \left( \prod_{j \neq i, k} F_j(x) \right) dx &= \int_{\mathbb{R}} f_k(x) F_i(x) \frac{d}{dx} \left( \prod_{j \neq i, k} F_j(x) \right) dx \\ \Leftrightarrow \int_{\mathbb{R}} \left( \frac{f_i(x)}{F_i(x)} - \frac{f_k(x)}{F_k(x)} \right) F_i(x) F_k(x) \frac{d}{dx} \left( \prod_{j \neq i, k} F_j(x) \right) dx &= 0.\end{aligned}\quad (\text{A.3})$$

Since (A.3) generally does not hold for all  $i, k \in \{1, \dots, n\}, i \neq k$  when the skill distributions of the competing players are distinct, we conclude that a symmetric equilibrium generally does not exist in the case of  $n > 2$  players. An exception is the “superstar” example we present in the next section.

## A.7 “Superstar” example

Proposition A.1 below illustrates a special case where  $n - 1$  homogeneous players compete against a player who is more highly skilled, giving rise to a symmetric equilibrium with  $n > 2$ . The key to understanding the example is that the reflected exponential distribution has a constant reversed hazard rate.

**Proposition A.1.** *Suppose that  $\Theta_i = t_i + \mathcal{E}_i$  for  $i = 1, \dots, n$  where  $t_1 > t_2 = \dots = t_n = t$  and the  $\mathcal{E}_i$ ,  $i = 1, \dots, n$ , are i.i.d. according to the reflected exponential distribution with cdf  $H(x) = \exp(\lambda x)$  defined on  $(-\infty, 0]$  with  $\lambda > 0$  (Rinne 2014). Then, a symmetric equilibrium exists in which all players choose the same equilibrium effort  $e^*$ .*

*Proof.* Player  $i$  outperforms player  $k$  iff

$$\begin{aligned} g_{e_i}(t_i + \varepsilon_i) &> g_{e_k}(t_k + \varepsilon_k) \\ \Leftrightarrow \varepsilon_k &< g_{e_k}^{-1}(g_{e_i}(t_i + \varepsilon_i)) - t_k. \end{aligned}$$

Recall that the  $\mathcal{E}_i$  are i.i.d., following the reflected exponential distribution on  $(-\infty, 0]$ . The cdf is denoted by  $H$  and the pdf by  $h$ . Hence, player  $i$  wins the contest with probability

$$\int \prod_{k \neq i} H(g_{e_k}^{-1}(g_{e_i}(t_i + x)) - t_k) h(x) dx.$$

In a symmetric equilibrium with  $e_1^* = \dots = e_n^* =: e^*$ , the marginal effect of effort on the probability of winning,

$$\int \left( \prod_{k \neq i} H(t_i + x - t_k) \right) \left( \sum_{k \neq i} \left( \frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(t_i + x)) \right) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(t_i + x - t_k)}{H(t_i + x - t_k)} h(x) dx,$$

must be the same for all  $i$ . Denote  $\Delta t = t_1 - t > 0$ . For player 1, we have,

$$\int \left( \prod_{k \neq 1} H(\Delta t + x) \right) \left( \sum_{k \neq 1} \left( \frac{d}{de_1} g_{e_k}^{-1}(g_{e_1}(t_1 + x)) \right) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} h(x) dx.$$

For any other player  $i \in \{2, \dots, n\}$ , we have

$$\begin{aligned} &\int \left( H(-\Delta t + y) \prod_{k \neq 1, i} H(y) \right) \left( \left( \frac{d}{de_i} g_{e_1}^{-1}(g_{e_i}(t + y)) \right) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(-\Delta t + y)}{H(-\Delta t + y)} \\ &+ \sum_{k \neq 1, i} \left( \frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(t + y)) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(y)}{H(y)} h(y) dy. \end{aligned}$$

The map  $\phi_1 : \mathbb{R}_x \rightarrow \mathbb{R}_y$  given by  $x \rightarrow y = \Delta t + x$  is a smooth diffeomorphism with  $\det |\phi_1'(x)| =$

1. Applying the associated change of variables to the preceding expression, we obtain

$$\begin{aligned} &\int \left( H(x) \prod_{k \neq 1, i} H(\Delta t + x) \right) \left( \left( \frac{d}{de_i} g_{e_1}^{-1}(g_{e_i}(t_1 + x)) \right) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(x)}{H(x)} \\ &+ \sum_{k \neq 1, i} \left( \frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(t_1 + x)) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} h(\Delta t + x) dx. \end{aligned}$$

The expressions for the two types of players can be restated as

$$\begin{aligned} & \int \left( H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left( \sum_{k \neq 1} \left( \frac{d}{de_1} g_{e_k}^{-1}(g_{e_1}(t_1 + x)) \right) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \frac{h(x)}{H(x)} dx, \\ & \int \left( H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left( \left( \frac{d}{de_i} g_{e_1}^{-1}(g_{e_i}(t_1 + x)) \right) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(x)}{H(x)} \\ & + \sum_{k \neq 1, i} \left( \frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(t_1 + x)) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \frac{h(\Delta t + x)}{H(\Delta t + x)} dx. \end{aligned}$$

Notice both expressions are equal to zero for  $x \geq -\Delta t$ . Hence, they can be restated as

$$\begin{aligned} & \int^{-\Delta t} \left( H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left( \sum_{k \neq 1} \left( \frac{d}{de_1} g_{e_k}^{-1}(g_{e_1}(t_1 + x)) \right) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \frac{h(x)}{H(x)} dx, \\ & \int^{-\Delta t} \left( H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left( \left( \frac{d}{de_i} g_{e_1}^{-1}(g_{e_i}(t_1 + x)) \right) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(x)}{H(x)} \\ & + \sum_{k \neq 1, i} \left( \frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(t_1 + x)) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} \frac{h(\Delta t + x)}{H(\Delta t + x)} \frac{h(\Delta t + x)}{H(\Delta t + x)} dx. \end{aligned}$$

For  $x < -\Delta t$ , we observe  $\frac{h(x)}{H(x)} = \frac{h(\Delta t + x)}{H(\Delta t + x)} = \lambda$ , and the expressions become

$$\begin{aligned} & \lambda^2 \int^{-\Delta t} \left( H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left( \sum_{k \neq 1} \left( \frac{d}{de_1} g_{e_k}^{-1}(g_{e_1}(t_1 + x)) \right) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} dx, \\ & \lambda^2 \int^{-\Delta t} \left( H(x) \prod_{k \neq 1} H(\Delta t + x) \right) \left( \sum_{k \neq i} \left( \frac{d}{de_i} g_{e_k}^{-1}(g_{e_i}(t_1 + x)) \right) \right) \Big|_{e_1^* = \dots = e_n^* = e^*} dx \end{aligned}$$

which are identical. □

## A.8 Heterogeneity in production functions, prizes and cost functions

In our previous analysis, the symmetry of the equilibrium was derived under the assumption that the production functions, prizes and cost functions were the same for the competing players. We show next that these assumptions can, under stricter assumptions, be relaxed.

We begin with the possibility to allow for heterogeneity in production technologies, using the observation that, in some situations, different production functions can be reinterpreted as different skill distributions. This allows us to show that a symmetric equilibrium exists using the results of Theorem 1.

**Corollary 3.** *Suppose that the production functions are different for the two competing*



players and can be written as  $g_i(\theta_i, e_i) = \tilde{g}(h_i(\theta_i), e_i)$ ,  $i \in \{1, 2\}$ . Then a symmetric equilibrium of the contest game exists with effort determined by

$$V \int_{\mathbb{R}} a_{e^*}(x) \tilde{f}_i(x) \tilde{f}_k(x) dx = c'(e^*), \quad (\text{A.4})$$

where  $h_i$  is a real-valued function,  $\tilde{f}_i$  denotes the pdf of the random variable  $\tilde{\Theta}_i := h_i(\Theta_i)$  and  $a_{e^*}$  is calculated based on the production function  $\tilde{g}(\tilde{\theta}_i, e_i)$ .

*Proof.* Define  $\tilde{\Theta}_i := h_i(\Theta_i)$ . The considered contest is then equivalent (in terms of equilibrium effort choices) to a contest in which skills are given by  $\tilde{\Theta}_i$  and the production function  $\tilde{g}(\tilde{\theta}_i, e_i)$  is the same for both players. Hence, a symmetric equilibrium exists with effort determined by the condition (A.4).  $\square$

As a specific example, consider the additive production function  $g_i(\theta_i, e_i) = \eta_i(\theta_i) + \kappa(e_i)$ , with  $\eta_i$  and  $\kappa$  being two strictly increasing functions. In this case, Corollary 3 can be applied since we can write  $\tilde{g}(h_i(\theta_i), e_i) = h_i(\theta_i) + \kappa(e_i)$ , with  $h_i(\theta_i) = \eta_i(\theta_i)$ . Likewise, in the case of a multiplicative production function of the form  $g_i(\theta_i, e_i) = \eta_i(\theta_i)\kappa(e_i)$  (imposing the additional assumption that  $\eta_i$  and  $\kappa$  are non-negative), we can apply the corollary noting that  $\tilde{g}(h_i(\theta_i), e_i) = h_i(\theta_i)\kappa(e_i)$ , with  $h_i(\theta_i) = \eta_i(\theta_i)$ . Notice that in both examples, the component  $\kappa(e_i)$  is the same for both players.

Next, we consider a situation with heterogeneous prizes given by  $V_1 = sV$  and  $V_2 = V$  with  $s > 0$ .

**Proposition A.2.** *Consider a contest with heterogeneous prizes  $V_1 = sV$  and  $V_2 = V$  where  $s > 0$ , and let  $c(e) = ce^\delta$ , with  $c > 0$ . Then, the contest can be transformed into a contest where the players have the same prizes, but different production functions by considering the transformed effort variables  $\xi_1 = e_1/s^{1/\delta}$  and  $\xi_2 = e_2$ . Denote the equilibrium of the transformed contest by  $\xi_1^*$  and  $\xi_2^*$ . Then, the equilibrium of the original contest is given by  $e_1^* = s^{1/\delta}\xi_1^*$  and  $e_2^*$ , where  $\xi_1^*$  and  $e_2^*$  maximize*

$$\int_{\mathbb{R}} F_2(g_{e_2}^{-1}(g_{s^{1/\delta}\xi_1}(x))) f_1(x) dx V - c(\xi_1)$$

and

$$\int_{\mathbb{R}} F_1(g_{s^{1/\delta}\xi_1}^{-1}(g_{e_2}(x))) f_2(x) dx V - c(e_2),$$

respectively.

*Proof.* Player 1's objective function is given by

$$\int_{\mathbb{R}} F_2(g_{e_2}^{-1}(g_{e_1}(x))) f_1(x) dx sV - c(e_1).$$

Using  $\xi_1 = e_1/s^{1/\delta}$ , the preceding expression becomes

$$\begin{aligned} & \int_{\mathbb{R}} F_2(g_{e_2}^{-1}(g_{s^{1/\delta}\xi_1}(x))) f_1(x) dx sV - c(s^{1/\delta}\xi_1) \\ &= \int_{\mathbb{R}} F_2(g_{e_2}^{-1}(g_{s^{1/\delta}\xi_1}(x))) f_1(x) dx sV - sc(\xi_1) \\ &= s \left( \int_{\mathbb{R}} F_2(g_{e_2}^{-1}(g_{s^{1/\delta}\xi_1}(x))) f_1(x) dx V - c(\xi_1) \right). \end{aligned}$$

Maximizing this function is equivalent to maximizing

$$\int_{\mathbb{R}} F_2(g_{e_2}^{-1}(g_{s^{1/\delta}\xi_1}(x))) f_1(x) dx V - c(\xi_1).$$

Player 2's objective function can be stated as

$$\begin{aligned} & \int_{\mathbb{R}} F_1(g_{e_1}^{-1}(g_{e_2}(x))) f_2(x) dx V - c(e_2) \\ &= \int_{\mathbb{R}} F_1(g_{s^{1/\delta}\xi_1}^{-1}(g_{e_2}(x))) f_2(x) dx V - c(e_2). \end{aligned}$$

□

Proposition A.2 shows that, for the given cost function, the equilibrium of a contest with heterogeneous prizes is characterized by conditions abiding a structure which is very similar to the structure of the conditions used to characterize the equilibrium in Theorem 1, although the equilibrium is in general no longer symmetric. However, as we shall see below, if the conditions of Corollary 3 are satisfied, we can derive a very simple expression for the relationship between the equilibrium effort levels. Before turning to this result, we first demonstrate that, in some situations, a contest with different cost functions is equivalent to a contest with different prizes (in terms of equilibrium effort choices) and hence can, according to Proposition A.2, be transformed into a contest with different production functions. This is formalized in the following remark.<sup>20</sup>

**Remark 1.** *Suppose that players have heterogeneous cost functions that take the form*

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<sup>20</sup>Similar transformations between prizes and cost functions are standard in the literature.

$c_i(e_i) = \omega_i c(e_i)$ , with  $\omega_i > 0$ . Then, the objective of player  $i$  can be written as:

$$\int_{\mathbb{R}} F_k(g_{e_k}^{-1}(g_{e_i}(x))) f_i(x) dx V_i - \omega_i c(e_i) = \omega_i \left( \int_{\mathbb{R}} F_k(g_{e_k}^{-1}(g_{e_i}(x))) f_i(x) dx \frac{V_i}{\omega_i} - c(e_i) \right).$$

This objective is equivalent to one in which prizes are given by  $\frac{V_i}{\omega_i}$ , but the cost functions are the same for both players.

Proposition A.2 and Remark 1 highlight how, in certain cases, contests with different prizes or different cost functions can be transformed into equivalent contests with different production technologies. We now turn to showing that, if the assumptions underlying Corollary 3 are satisfied, these contests can be reinterpreted as contests with different skill distributions, allowing us to apply the equilibrium characterization from our baseline case.

Proposition A.3 below considers the Cobb-Douglas production technology that satisfies the assumptions of Corollary 3. We can show that the ratio of equilibrium efforts  $e_1^*/e_2^*$  depends only on the ratio of prizes and the degree of homogeneity of the cost function.

**Proposition A.3.** *Let the two prizes be given by  $V_1 = sV$  and  $V_2 = V$  with  $s > 0$ . Suppose  $g(\theta, e) = \theta^\alpha e^\beta$ , with  $\alpha, \beta > 0$ , and let  $c(e) = ce^\delta$ , with  $c > 0$ . Then an equilibrium exists with efforts given by  $e_1^* = s^{1/\delta} e_2^*$  and  $e_2^*$  being determined by*

$$\int_{\mathbb{R}} \tilde{f}_1(x) \tilde{f}_2(x) x dx V = e_2^* c'(e_2^*),$$

where  $\tilde{f}_1$  and  $\tilde{f}_2$  denote the pdfs of the random variables  $\tilde{\Theta}_1 := s^{1/\delta} \Theta_1^{\alpha/\beta}$  and  $\tilde{\Theta}_2 := \Theta_2^{\alpha/\beta}$ , respectively.

*Proof.* Player 1 wins if and only if

$$\begin{aligned} \theta_1^\alpha e_1^\beta &> \theta_2^\alpha e_2^\beta \\ \Leftrightarrow \theta_1^{\frac{\alpha}{\beta}} e_1 &> \theta_2^{\frac{\alpha}{\beta}} e_2. \end{aligned}$$

Substituting  $\xi_1 = e_1/s^{1/\delta}$ , the condition becomes

$$\theta_1^{\frac{\alpha}{\beta}} s^{1/\delta} \xi_1 > \theta_2^{\frac{\alpha}{\beta}} e_2.$$

Now define  $\tilde{\Theta}_1 := s^{1/\delta} \Theta_1^{\alpha/\beta}$  and  $\tilde{\Theta}_2 := \Theta_2^{\alpha/\beta}$ , and denote the corresponding pdfs and cdfs by  $\tilde{f}_1$ ,  $\tilde{f}_2$ ,  $\tilde{F}_1$ , and  $\tilde{F}_2$ , respectively.

Player 1's objective function can be stated as

$$\begin{aligned} & \int_{\mathbb{R}} \tilde{F}_2\left(\frac{\xi_1 x}{e_2}\right) \tilde{f}_1(x) dx sV - c\left(s^{1/\delta} \xi_1\right) \\ &= s \left( \int_{\mathbb{R}} \tilde{F}_2\left(\frac{\xi_1 x}{e_2}\right) \tilde{f}_1(x) dx V - c(\xi_1) \right). \end{aligned}$$

Maximization of the objective function is equivalent to maximization of

$$\int_{\mathbb{R}} \tilde{F}_2\left(\frac{\xi_1 x}{e_2}\right) \tilde{f}_1(x) dx V - c(\xi_1),$$

so we consider this latter problem. Since player 2's objective function can be stated as

$$\int_{\mathbb{R}} \tilde{F}_1\left(\frac{e_2 x}{\xi_1}\right) \tilde{f}_2(x) dx V - c(e_2),$$

we have transformed the contest into the form of our main model, meaning that an equilibrium  $(e_1^*, e_2^*)$  exists, where  $e_2^* = \xi_1^* = \frac{e_1^*}{s^{1/\delta}}$  is characterized by

$$\int_{\mathbb{R}} \tilde{f}_1(x) \tilde{f}_2(x) x dx V = e_2^* c'(e_2^*).$$

□

We conclude this section by presenting an example in which players, in addition to having different skill distributions as in our baseline case, have different production functions and cost functions, and face different prizes. This example is equivalent to a Tullock lottery contest with heterogeneous prizes and quadratic effort costs.

**Example 5.** *Suppose that:*

$$\begin{aligned} g_1(\theta_1, e_1) &= \frac{\theta_1 e_1}{2}, & c_1(e_1) &= e_1^2, & V_1 &= \frac{V}{2}, & f_1(x) &= 2 \frac{\exp(-2x^{-1})}{x^2} I_{\{x>0\}}, \\ g_2(\theta_2, e_2) &= \theta_2 e_2, & c_2(e_2) &= \frac{e_2^2}{2}, & V_2 &= V, & f_2(x) &= \frac{\exp(-x^{-1})}{x^2} I_{\{x>0\}}, \end{aligned}$$

implying that player 2 has a more efficient production technology, lower cost of exerting effort and faces a higher prize. Define  $\tilde{\Theta}_1 := \frac{\Theta_1}{2}$ , and denote by  $\tilde{f}_1(x) = \frac{\exp(-x^{-1})}{x^2} I_{\{x>0\}}$  and  $\tilde{F}_1(t) = \int_{-\infty}^t \frac{\exp(-x^{-1})}{x^2} I_{\{x>0\}} dx$  the corresponding pdf and cdf.

The objective function of player 1 can then be stated as:

$$\int_{\mathbb{R}} F_2\left(\frac{e_1 x}{e_2}\right) \tilde{f}_1(x) dx \frac{V}{2} - e_1^2 = 2 \left( \int_{\mathbb{R}} F_2\left(\frac{e_1 x}{e_2}\right) \tilde{f}_1(x) dx \frac{V}{4} - \frac{e_1^2}{2} \right).$$

The objective function of player 2 can be stated as:

$$\int_{\mathbb{R}} \bar{F}_1\left(\frac{e_2 x}{e_1}\right) f_2(x) dx V - \frac{e_2^2}{2}.$$

Hence, we have transformed the original contest into a contest with different prizes  $\tilde{V}_1 := \frac{V}{4}$  and  $V_2 = V$ , but identical production functions and cost functions. According to part (i) of Proposition A.3 (noting that  $s = \frac{1}{4}$  and  $\delta = 2$  in the transformed contest), an equilibrium exists with efforts given by  $e_1^* = \frac{e_2^*}{\sqrt{4}} = \frac{e_2^*}{2}$  where  $e_2^*$  is determined by

$$V \int_{\mathbb{R}} \tilde{f}_1(x) f_2(x) x dx = (e_2^*)^2 \iff e_2^* = \sqrt{V \int_{\mathbb{R}} \tilde{f}_1(x) f_2(x) x dx},$$

with  $\tilde{f}_1(x) = \frac{\exp\left(-\frac{x^{-1}}{2}\right)}{2x^2} I_{\{x>0\}}$  being the pdf corresponding to the random variable  $\frac{\tilde{\Theta}_1}{2}$ . Using the specific density functions, we obtain

$$e_2^* = \sqrt{V \int_0^{\infty} \frac{\exp\left(-\frac{x^{-1}}{2}\right)}{2x^2} \frac{\exp(-x^{-1})}{x^2} x dx} = \sqrt{V \int_0^{\infty} \frac{\exp\left(-\frac{3}{2}x^{-1}\right)}{2x^3} dx} = \sqrt{\frac{2V}{9}}.$$

Notice that the same result as in Example 5 would be obtained by directly solving the Tullock contest with different prizes and quadratic costs, in which players maximize the objectives  $\frac{e_1}{e_1+e_2} \frac{V}{4} - \frac{e_1^2}{2}$  and  $\frac{e_2}{e_1+e_2} V - \frac{e_2^2}{2}$ , respectively.

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